HW #3

1) $\sum_{n=1}^{\infty} \frac{n^7}{7^n}$ We compute $\lim_{n\to\infty} \left| \frac{(n+1)^7/7^{n+1}}{n^7/7^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^7\, 7^n}{n^7\, 7^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^7}{n^7} \cdot \frac{1}{7} \right| = \frac{1}{7} \lim_{n\to\infty} \left| \frac{(n+1)^7}{n^7} \right|$$

$$= 1/7$$

so by the ratio test, $\sum_{n=1}^{\infty} n^7/7^n$ converges.

Using the root test,

$$\lim_{n\to\infty} \left(\frac{n^7}{7^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{7/n}}{7^{n/n}} = \frac{1}{7} \lim_{n\to\infty} n^{7/n} = 1/7$$

This also shows convergence. | as $\lim_{n\to\infty} n^{k/n} = 1$ for all fixed k.

2) $\sum_{n=1}^{\infty} \arctan(e^{-n})$ We compute $\lim_{n\to\infty} \left| \frac{\arctan(e^{-n-1})}{\arctan(e^{-n})} \right|$

view as a function and use L'Hopital

$$\overset{?}{=} \lim_{n\to\infty} \left| \frac{-e^{-n-1}/1+e^{-2n-2}}{-e^{-n}/1+e^{-2n}} \right| = \lim_{n\to\infty} \left| \frac{e^{-n-1}}{e^{-n}} \cdot \frac{1+e^{-2n}}{1+e^{-2n-2}} \right|$$

$$= \lim_{n\to\infty} \frac{1}{e} \left| \frac{1+e^{-2n}}{1+e^{-2n-2}} \right| = \frac{1}{e} \cdot \frac{1+1}{1+1} = \frac{1}{e}.$$

Here, we use $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

and $\lim_{n\to\infty} e^{-2n} = 1$

so by the ratio test, $\sum_{n=1}^{\infty} \arctan(e^{-n})$ converges.

<u>Fact for #8</u>

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{x}{n}\right)^n\right)} = \lim_{n\to\infty} e^{n\ln\left(1+\frac{x}{n}\right)} = \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)/(1/n)}$$

$$= e^{\lim_{n\to\infty} \ln(1+x/n)/(1/n)} \underset{\text{L'Hopital}}{=} e^{\lim_{n\to\infty} -\frac{x}{n^2}\frac{1}{1+x/n}/\left(-\frac{1}{n^2}\right)} = e^{\lim_{n\to\infty} \frac{x}{1+x/n}}$$

$$= e^x$$

$$\boxed{\text{so } \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x}$$

3) $\sum_{n=1}^{\infty} \frac{3^{2n+1}}{(2n+1)!}$

converges by the ratio test.

We compute

$$\lim_{n\to\infty} \left| \frac{3^{2(n+1)+1}/(2(n+1)+1)!}{3^{2n+1}/(2n+1)!} \right| = \lim_{n\to\infty} \left| \frac{3^{2n+3}(2n+1)!}{3^{2n+1}(2n+3)!} \right|$$

$$= \lim_{n\to\infty} \frac{9}{(2n+3)(2n+2)} = 0$$

Alternatively, note

$$\sum_{n=1}^{\infty} \frac{3^{2n+1}}{(2n+1)!} \le \sum_{n=0}^{\infty} 3^n/n! = e^3.$$

Since $\left\{ \sum_{n=1}^{k} \frac{3^{2n+1}}{(2n+1)!} \right\}_{k=1}^{\infty}$ is an increasing, bounded sequence it converges.

4) $1 + \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} + \ldots = 1 + \frac{1}{2} + \frac{1}{2\cdot 4} + \ldots$ so $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{2(n+1)\cdot 2n\cdots 2}}{\frac{1}{(2n)\cdot 2(n-1)\cdots 2}} \right| = \lim_{n\to\infty} \frac{1}{2(n+1)} = 0$

converges by the ratio test

5) $\sum_{n=2}^{\infty} 1/\ln(n)$

$1/n \le 1/\ln(n)$ so since $\sum_{n=2}^{\infty} 1/n$ diverges,

by the squeeze theorem $\sum_{n=2}^{\infty} 1/\ln(n)$ diverges.

6) $\sum_{n=1}^{\infty} \left(\frac{2n^2+3}{3n^2+2}\right)^{5n}$

converges by the root test.

We compute

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^2+3}{3n^2+2}\right)^{5n}} = \lim_{n\to\infty} \left(\frac{2n^2+3}{3n^2+2}\right)^{5n/n}$$

$$= \lim_{n\to\infty} \left(\frac{2n^2+3}{3n^2+2}\right)^5 = \left(\lim_{n\to\infty} \frac{2n^2+3}{3n^2+2}\right)^5 = \left(\frac{2}{3}\right)^5$$

7) $\sum_{n=1}^{\infty} n!/n^n$

converges by the ratio test

We compute

$$\lim_{n\to\infty} \left| \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\, n^n}{(n+1)(n+1)^n} \right| = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right| = \lim_{n\to\infty} \left| 1 \Big/ \frac{(n+1)^n}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| 1 \Big/ \left(\frac{n+1}{n}\right)^n \right| = \lim_{n\to\infty} \left| 1 \Big/ \left(1+\frac{1}{n}\right)^n \right| = 1/e$$

8) $\sum_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)^{n^2}$ diverges to ∞ by the root test

We compute

$$\lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{2n}\right)^{n^2}} = \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n^2/n}$$

$$= \lim_{n\to\infty}\left(1+\frac{\left(\frac{1}{2}\right)}{n}\right)^n = e^{1/2}$$

see page 1

Alternatively, $\left(1+\frac{1}{2^n}\right)^{n^2} > 1$, so $\sum_{n=1}^{\infty}\left(1+\frac{1}{2^n}\right)^{n^2} > \sum_{n=1}^{\infty} 1$ which diverges to ∞

Therefore, by the Squeeze theorem $\sum_{n=1}^{\infty}\left(1+\frac{1}{2^n}\right)^{n^2}$ diverges to ∞.

9) $\sum_{n=1}^{\infty}\left(\sqrt{4n+1}\sqrt{n}-2n\right)^n = \sum_{n=1}^{\infty}\left(\sqrt{n}\left(\sqrt{4n+1}-2\sqrt{n}\right)\right)^n$

We compute $\left(\sqrt{4n+1}-\sqrt{4n}\right)\frac{\left(\sqrt{4n+1}+\sqrt{4n}\right)}{\left(\sqrt{4n+1}+\sqrt{4n}\right)} = \frac{4n+1-4n}{\sqrt{4n+1}+\sqrt{4n}} = \frac{1}{\sqrt{4n+1}+\sqrt{4n}}$

Then $\lim_{n\to\infty}\sqrt[n]{\left(\sqrt{4n+1}\sqrt{n}-2n\right)^n} = \lim_{n\to\infty}\sqrt{4n+1}\sqrt{n}-2n = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{4n+1}+\sqrt{4n}}$

$$= \lim_{n\to\infty}\frac{\sqrt{n}/\sqrt{n}}{\sqrt{4n+1/n}+\sqrt{4n/n}} = \lim_{n\to\infty}\frac{1}{\sqrt{4+\frac{1}{n}}+\sqrt{4}} = \frac{1}{2+2} = 1/4$$

so $\sum_{n=1}^{\infty}\left(\sqrt{4n+1}\sqrt{n}-2n\right)^n$ converges by the root test.

#2

8) $\sum_{n=1}^{\infty} x^n/n!$ converges by the ratio test

We compute $\lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 \text{ for all } x$$

#2 9) $\sum_{n=1}^{\infty} n^n x^n$ converges for $x=0$, else diverges by root test

We compute $\lim_{n\to\infty}\sqrt[n]{n^n x^n} = \lim_{n\to\infty}(nx)^{n/n} = \lim_{n\to\infty} nx$

$$= \begin{cases} 0 & x=0 \\ \text{diverges} & x>0 \end{cases}$$

If $x\neq 0$, for $n > 1/x$, see $n^n x^n > \left(\frac{1}{x}\right)^n x^n = 1$, so diverges by squeeze theorem